Working paper: 2-p-opt local search in the Probabilistic Traveling Salesman Problem (PTSP)

Leonora Bianchi -IDSIA -

May 15, 2001

Abstract

We consider the 2-opt local search applied to the PTSP (2-p-opt). We write the equation for the variation in the PTSP objective function when a 2-p-opt move is performed on a tour. Our result revises the equations formerly derived by Bertsimas, and extend them to the general case of both symmetric and asymmetric PTSP.

In the following we first give the expression for the variation of the objective function (chapter 1). At this stage the 2-p-opt local search seems to require $O(n^3)$ computation time. An open question is whether the expression derived for the variation of the objective function can be put in a recursive form. In case this is possible, the 2-p-opt local search would only require $O(n^2)$ computation time. This open question is discussed in chapter 2.

Finally, in chapter 3, we discuss the recursive equations derived by Bertsimas in [Ber88] for the 2-p-opt local search in the case of symmetric PTSP, and we show that they are not correct.

1 Variation of the objective function with a 2-p-opt move

We consider the PTSP where each city $i = 1, 2, ..., n$ has a probability p of requiring a visit independently from other cities. We focus on the 2-opt local search, which is called 2-p-opt local search in the context of the PTSP. Given a tour τ, its 2-p-opt neighborhood is the set of tours $\tilde{\tau}$ obtained by reversing a section of τ and adjusting the arcs adjacent to this section, like in fig.1. The PTSP objective function is the expected length of a tour over all possible instance realizations under the 'skipping strategy' [Jai85]. When $\tau = (1, 2, ..., n)$,

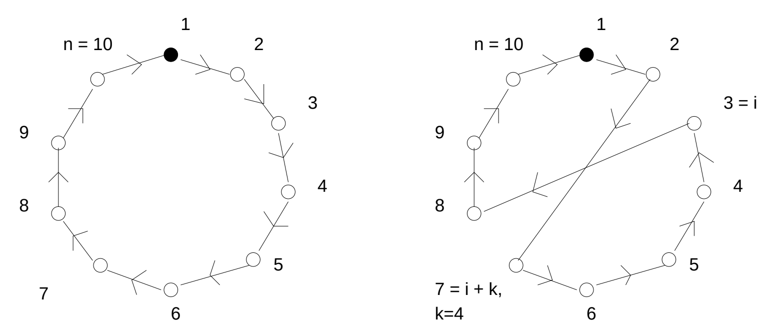

Figure 1: Tour τ (left) and tour $\tilde{\tau}$ (right) obtained by reversing the section $(i, ..., i + k)$

the expected length is

$$E[L(\tau^p)] = \sum_{j=1}^{n}\sum_{r=1}^{n-1} p^2(1-p)^{r-1} d(j,(j+r)\mathrm{mod} n). \tag{1}$$

In equation (1) the distance between the j^{th} city and the $(j+r)^{\text{th}}$ city is weighted by the probability that the two nodes require a visit while the nodes between them do not require it, that is $p^2(1-p)^{r-1}$. In general, if $\tau(j)$ is the j^{th} city of tour τ, for every couple of nodes $\tau(j),\ \tau(l)$ we must weight the corresponding distances by the proper probability factors

$$\begin{array}{lll} l > j & \Rightarrow & d(\tau(j),\tau(l))\ p^2(1-p)^{l-j-1} \\ l < j & \Rightarrow & d(\tau(j),\tau(l))\ p^2(1-p)^{n-(j-l)-1}. \end{array} \tag{2}$$

When we consider a tour $\tilde{\tau}$ of the 2-p-opt neighborhood of τ, the reversing of section $(i,...,i+k)$ causes some couples of nodes to have in $\tilde{\tau}$ probability factors different from those in τ. This is at the core of the change in the expected length $E[L(\tau^p)] - E[L(\tilde{\tau}^p)] = \Delta E_{i,k}$.

Before writing $\Delta E_{i,k}$, we introduce the following three-dimensional matrices of partial results (to be computed in the first phase of the 2-p-opt local search)

$$A_{i,k,l} = \sum_{r=k}^{l}(1-p)^{r-1}d(i,i+r) \quad \text{and} \quad B_{i,k,l} = \sum_{r=k}^{l}(1-p)^{r-1}d(i-r,i) \tag{3}$$

with $1 \leq k \leq n-1$ and $1 \leq l \leq n-1$, and where we assume that

$$i \pm r = \begin{cases} (i \pm r)\mathrm{mod} n, & \text{if } i \pm r \neq 0 \text{ and if } i \pm r \text{ is not a multiple of } n \\ n, & \text{otherwise}\ . \end{cases} \tag{4}$$

Matrices A and B are directly related to the objective function computation of the tour τ, $E[L(\tau^p)]$. They contain partial sums of arc costs weighted by the proper probability factors appearing in $E[L(\tau^p)]$, as is easy to see by comparing (3) with (1).

Let us now consider a tour $\tilde{\tau}$ obtained from τ by reversing a section $(i,i+1,...,i+k)$ of it, like in fig.1. In calculating $\Delta E_{i,k}$ we must consider all arcs of the complete graph over the n cities, and multiply them by the difference in probability factors due to the reversing of the section. We refer to the terms involving the couple of nodes (j,k) as the *interaction* between j and k, and we label the first (i) and the last $(i+k)$ node of the reversed section respectively a and b. In order to take into account all the arcs of the complete graph, we must consider then the interaction between all nodes, precisely the interaction between a and b, between a, b and the outside of the reversed section, between a, b and the inside of the reversed section, between the outside and the inside of the reversed section, and in the inside, that is, with obvious notation

$$\Delta E_{i,k} = p^2[Int\,ab_{i,k} + IntOut\,a_{i,k} + IntOut\,b_{i,k} + IntIn\,a_{i,k} + IntIn\,b_{i,k} + IntInOut_{i,k} + IntInIn_{i,k}]. \tag{5}$$

The terms in (5) have the following form

$$IntOut\,a_{i,k} = [q^{-k}-1]A_{i,k+1,n-1} + [q^{k}-1]B_{i,1,n-k-1} \tag{6}$$

$$IntOut\,b_{i,k} = [q^{k}-1]A_{i+k,1,n-k-1} + [q^{-k}-1]B_{i+k,k+1,n-1} \tag{7}$$

$$IntIn\,a_{i,k} = \sum_{l=1}^{k-1}[q^{n-2l}-1]A_{i,l,l} + \sum_{l=n-k+1}^{n-1}[q^{n-2l}-1]B_{i,l,l} \tag{8}$$

$$IntIn\,b_{i,k} = \sum_{l=1}^{k-1}[q^{n-2l}-1]B_{i+k,l,l} + \sum_{l=n-k+1}^{n-1}[q^{n-2l}-1]A_{i+k,l,l} \tag{9}$$

$$Int\,ab_{i,k} = [q^{n-2k}-1]A_{i,k,k} + [q^{-n+2k}-1]B_{i,n-k,n-k} \tag{10}$$

$$IntInIn_{i,k} = \sum_{l=1}^{k-2} IntIn\,a_{i+l,k-l} \tag{11}$$

$$IntInOut_{i,k} = \sum_{l=1}^{k-1} IntOut\,[a+l]_{i,k} \tag{12}$$

where

$$IntOut\,[a+l]_{i,k} = [q^{2l-k}-1]A_{i+l,k+1-l,n-1-l} + [q^{-2l+k}-1]B_{i+l,l+1,n-k-1+l}. \tag{13}$$

2 Recursive equations

A few comments are in order about the computation complexity of the above equations. Let us suppose we have computed in advance the matrices A and B. Then the interactions $IntOut\,a_{i,k}$ and $IntOut\,b_{i,k}$ are of complexity $O(1)$, and the other ones are of complexity $O(k)$. Therefore checking all the 2-p-opt neighborhood of tour τ require with the above equations $O(n \cdot \sum_{k=1}^{n} k) = O(n \cdot n^2) = O(n^3)$.

In fact one should not stop here, but instead investigate whether it is possible to express the interactions due to the reversing a k-node section in terms of interactions due to the reversing of $(k-1)$-node sections. The advantage of having such a recursion is that than it is possible to explore the neighborhood by increasing k, exploiting at each k previously computed (i.e. with smaller k) arrays of information. This would keep the 2-p-opt local search complexity of order $O(n^2)$.

Up to now we found that it is possible to express recursively all interactions, except $IntInOut_{i,k}$. The question whether a recursive equation for this interaction exists is still open, we are working on it.

In the following we report the recursive relations we found.

$$\begin{aligned} IntIn\,a_{i,k} &= IntIn\,a_{i,k-1} + [q^{n-2(k-1)}-1]A_{i,k-1,k-1} + [q^{-n+2(k-1)}-1]B_{i,n-k+1,n-k+1} && (14)\\ IntIn\,b_{i,k} &= IntIn\,b_{i+1,k-1} + [q^{n-2(k-1)}-1]B_{i+k,k-1,k-1} + [q^{-n+2(k-1)}-1]A_{i+k,n-k+1,n-k+1} && (15)\\ IntInIn_{i,k} &= IntInIn_{i+1,k-1} + IntIn\,a_{i+1,k-1} && (16) \end{aligned}$$

3 Bertsimas recursive equation for the symmetric PTSP

Bertsimas in [Ber88, BJO90, BH93] studies the 2-p-opt local search in the case of symmetric PTSP. He writes a recursive equation for $\Delta E_{i,k}$ as a function of $\Delta E_{i+1,k-1}$ and other quantities. Checking the 2-p-opt neighborhood of a tour τ requires then two 'phases'. In the first phase one computes $\Delta E_{i,1}$ for every value of i. These n calculations require $O(n)$ time apiece, and $O(n^2)$ time in all. During this phase some matrices of information are stored, which will be used later. In the second phase one computes recursively the $\Delta E_{i,k}$ for every i, by increasing k, $k = 2, 3, ..., n-1$. Each $\Delta E_{i,k}$ is now computable in $O(1)$, thanks to the recursion and to the information stored in the first phase. Therefore all the process of checking the 2-p-opt neighborhood is done in $O(n^2)$ time. Here we show that Bertsimas recursive equation for $\Delta E_{i,k}$ is not actually correct.

According to Bertsimas[1] we have, for $k = 1, 2, ..., n-1$ and $i = 1, 2, ..., n$

$$\begin{cases} \Delta E_{i,k} = \Delta E_{i+1,k-1} + p^2[IntOut\,a_{i,k} + IntOut\,b_{i,k} + IntIn\,a_{i,k} + IntIn\,b_{i,k}] \\ \Delta E_{i,0} = IntIn\,a_{i,1} = IntIn\,b_{i,1} \equiv 0. \end{cases} \tag{17}$$

The terms inside squared brackets of equation (17) are the interactions between the first (a) and the last (b) node of the reversed section and the other parts of the tour (i.e. the 'inside' and the 'outside' of the

[1] For convenience we use a notation which is slightly different from Bertsimas one. The relation between the two notations is $A(i,k,l) = A_{i,k} - A_{i,l+1}$ and $B(i,k,l) = B_{i,k} - B_{i,l+1}$.

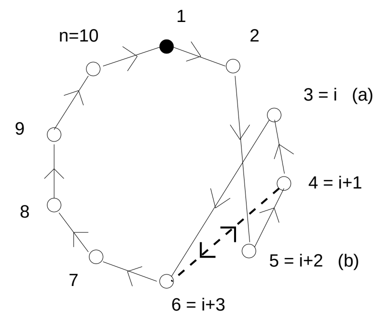

Figure 2: Tour $\tilde{\tau}$, with $i = \tilde{\tau}(i+2)$, $i+1 = \tilde{\tau}(i+1)$, $i+2 = \tilde{\tau}(i)$ and $i+3 = \tilde{\tau}(i+3)$. The dashed line represents arcs $(i+1, i+3)$ and $(i+3, i+1)$.

reversed section):

$$IntOut\, a_{i,k} = [q^{-k} - 1]A_{i,k+1,n-1} + [q^k - 1]B_{i,1,n-k-1} \tag{18}$$
$$IntOut\, b_{i,k} = [q^k - 1]A_{i+1,1,n-k-1} + [q^{-k} - 1]B_{i+k,k+1,n-1} \tag{19}$$

and

$$IntIn\, a_{i,k} = [q^{n-k} - 1]A_{i,1,k-1} + [q^{k-n} - 1]B_{i,n-k+1,n-1}$$
$$IntIn\, b_{i,k} = [q^{k-n} - 1]A_{i+1,n-k+1,n-1} + [q^{n-k} - 1]B_{i+k,1,k-1} \tag{20}$$

with $q = 1 - p$. If $k = 1$ equation (17) can be simplified to

$$\Delta E_{i,1} = p^3[q^{-1}A_{i,2,n-1} - B_{i,1,n-2} - A_{i+1,1,n-2} + q^{-1}B_{i+1,2,n-1}]. \tag{21}$$

Bertsimas expression for $\Delta E_{i,k}$ is correct for $k = 1$ (equation (21)), while it is not for $k \geq 2$.

Proposition 1 *Given a tour* $\tau = (1, 2, ..., n)$ *and a tour* $\tilde{\tau} = (1, 2, ..., i+k, i+k-1, ..., i, i+k+1, ...n)$ *with* $k \geq 2$*, then* $E[L(\tau^p)] - E[L(\tilde{\tau}^p)] \neq \Delta E_{i,k}$.

It is enough to show that the inequality holds for $k = 2$. In this case

$$\Delta E_{i,2} = \Delta E_{i+1,1} + p^2[IntOut\, a_{i,2} + IntOut\, b_{i,2} + IntIn\, a_{i,2} + IntIn\, b_{i,2}]. \tag{22}$$

We show that in this expression there are arcs with the wrong probability factor. Consider for instance arc $(i+1, i+3)$ and arc $(i+3, i+1)$. These arcs are only counted in the first term of (22) $\Delta E_{i+1,1}$, namely in $IntOut\, a_{i+1,1}$. If, by the help of equation (18), we isolate in $IntOut\, a_{i+1,1}$ the terms involving arcs $(i+1, i+3)$ and $(i+3, i+1)$ we get, according to Bertsimas

$$p^2[q^{-1} - 1]qd(i+1, i+3) \neq 0 \quad \text{and} \quad p^2[q-1]q^{n-3}d(i+3, i+1) \neq 0. \tag{23}$$

Now we consider the objective function (1), and we write the probability factors that arcs $(i+1, i+3)$ and $(i+3, i+1)$ must have, respectively in $E(L(\tau^p))$ and in $E(L(\tilde{\tau}^p))$. When arc $(i+1, i+3)$ belongs to τ it has a probability factor of p^2q, as it is easy to see by taking into account (2) and by writing $(i+1, i+3) = (i+1, (i+1)+2)$. The same factor is valid when the arc belongs to $\tilde{\tau}$, because $(i+1, i+3) = (\tilde{\tau}(i+1), \tilde{\tau}(i+3))$. Figure 2 may be of help in visualizing this. Similarly, arc $(i+3, i+1)$ has probability factor p^2q^{n-3} both in τ and in $\tilde{\tau}$ (since $i+3 = \tilde{\tau}(i+1-(n-2))$). Therefore the right overall factors for $(i+1, i+3)$ and $(i+3, i+1)$ in $\Delta E_{i,k} = E(\tilde{\tau}^p) - E(L(\tau^p))$ are

$$p^2[q - q]d(i+1, i+3) = 0 \quad \text{and} \quad p^2[q^{n-3} - q^{n-3}]d(i+3, i+1) = 0 \tag{24}$$

in contrast with (23). ■

References

[Ber88] Dimitris J. Bertsimas. *Probabilistic Combinatorial Optimization Problems.* PhD thesis, M.I.T., 1988.

[BH93] Dimitris J. Bertsimas and Louis Howell. Further results on the probabilistic traveling salesman problem. *European Journal of Operational Research*, 65:68–95, 1993.

[BJO90] Dimitris J. Bertsimas, Patrick Jaillet, and Amedeo Odoni. A priori optimization. *Operation Research*, 38:1019–1033, 1990.

[Jai85] Patrick Jaillet. *Probabilistic Traveling Salesman Problems.* PhD thesis, M.I.T., 1985.